# Experimental observation of soliton propagation and annihilation in a hydromechanical array of one-way coupled oscillators

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We have experimentally realized unidirectional or one-way coupling in a mechanical array by powering the coupling with flowing water. In cyclic arrays with an even number of elements, solitonlike waves spontaneously form but eventually annihilate in pairs, leaving a spatially alternating static attractor. In cyclic arrays with an odd number of elements, this alternating attractor is topologically impossible, and a single soliton always remains to propagate indefinitely. Our experiments with 14- and 15-element arrays highlight the dynamical importance of both noise and disorder and are further elucidated by our computer simulations.

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## I. INTRODUCTION

Unidirectional or one-way coupling enables the propagation of solitary waves or solitons [1] and serves as an extreme example of wave propagation in an anisotropic medium [2,3]. It is a fascinating paradigm for nonlinear dynamics, which has inspired much recent work [4–10].

In *et al.* [4] introduced the idea of one-way coupling as a way of improving the sensitivity and noise tolerance of fluxgate magnetometers. They established that cyclic arrays or rings of bistable but overdamped and undriven systems can oscillate if they are one-way coupled with sufficient strength. Lindner and Bulsara [8] understood the oscillations of such arrays as arising from a "frustrated" equilibrium, where waves of dislocations in equilibria propagate endlessly around rings of one-way coupled elements. Furthermore, they demonstrated how noise and coupling mediate the consequent complex spatiotemporal dynamics.

While the idea of one-way coupling emerged in the context of electronic sensors, and we are aware of the use of one-way coupling in the construction of ring oscillators in microelectronic circuits [11], we are unaware of any previous mechanical realization of the different dynamics afforded by one-way coupling. In this paper, we report the experimental observation of the propagation and annihilation of solitons [12] in relatively long (14- and 15-element) mechanical arrays of one-way coupled oscillators. We observe the spontaneous formation and subsequent annihilation of solitons, including the distributions of their lifetimes. We also observe the dramatic qualitative difference between the dynamics of arrays of even and odd length. Our experiments compare favorably with computer simulations that include both noise and disorder (temporal and spatial irregularities).

## II. THEORY

Consider a nonlinear oscillator located by equilibrium coordinate $\theta$ and confined by a bistable potential $V[\theta] = -a\theta^2/2 + b\theta^4/4$ with positive and negative stable equilibria $\theta_\pm = \pm\sqrt{a/b}$. Given rotational inertia $I$, viscosity $\gamma$, and coupling $k$, the two-way coupled equation of motion of oscillator $n$ in an array of $N$ such elements is

$$I\ddot{\theta}_n + \gamma\dot{\theta}_n = k(\theta_{n+1} - \theta_n) - k(\theta_n - \theta_{n-1}) + V'[\theta_n], \quad (1)$$

where the overdots represent differentiation with respect to time, and the prime indicates differentiation with respect to position. Formally, we achieve one-way coupling by deleting one of the two coupling terms to get

$$I\ddot{\theta}_n = -\gamma\dot{\theta}_n + A\theta_n - b\theta_n^3 - \kappa\theta_{n-1}, \quad (2)$$

where $A = a + \kappa$ and $\kappa = -k > 0$. The coupling both renormalizes the shape of the potential and inversely affects the next oscillator. Effectively, we add an external forcing term that negates the coupling term we deleted. One-way coupling is made possible by having each oscillator modify an external force on the next oscillator.

Such arrays come to rest in several ways. For example, all the oscillators can assume their positive stable states $\theta_n = \theta_+$, where the coupling and spring forces vanish. More generically, the oscillators can be paired in spatially alternating positive and negative positions, where the coupling forces balance the spring forces.

In finite arrays with periodic boundary conditions, the number of elements critically determines the qualitative dynamics, as illustrated by Fig. 1. In even arrays, pairs of solitons collide and annihilate as the array dissipates the kinetic energy of the back and forth motion of the oscillators. This leaves the array in a spatially alternating static state. In odd arrays, complete soliton pairing is impossible. The consequent topological frustration leaves a single soliton that propagates forever around the ring [8].

By allowing information to flow in only one direction, one-way coupling breaks the rotational symmetry of the ring. Arrays with one-way coupling are more than mere mathematical curiosities, and they can be realized so long as an external force powers the coupling.

## III. APPARATUS

Our experimental apparatus, shown in Fig. 2, is a linear array of elements with periodic boundary conditions. Each element is an inverted pendulum restrained by two springs that together form a bistable oscillator. The springs are *not*

 

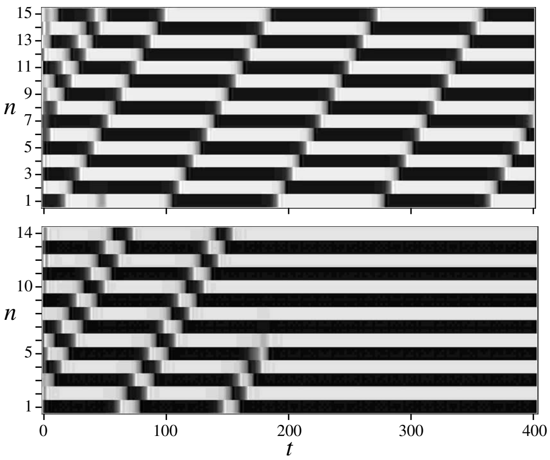


FIG. 1. Simulated spacetime evolution $\theta_n[t]$ of a cyclic array of $N=15$ (top) and $N=14$ (bottom) one-way coupled oscillators. Grays code oscillator angles. From random initial conditions, a static, spatially alternating state beckons where individual oscillators are at rest, rotated positively (white) or negatively (black). Solitons appear as diagonal discontinuities in this spatial alternation, and annihilate in pairs until one or zero remain, depending on the parity of the array.

involved in the coupling. Instead, each oscillator is linearly coupled to the next by flowing water. One-way coupling is achieved by mechanically directing this flow so that each oscillator causes the next to rotate about a central axle common to all the oscillators.

The individual oscillators, shown schematically in Fig. 3, are constructed primarily from machined aluminum combined with cut and cemented acrylic. The pendulum arms and base are made from 0.16-cm-thick aluminum. The arms are 25-cm-long strips and are attached to a 16.5 cm $\times$5.0 cm base. Compartmented boxes made of 0.16-cm-thick acrylic, installed on the base of the oscillator, exploit the water's weight to help reverse the oscillator. Holes drilled in the bottom of the compartments and ex-

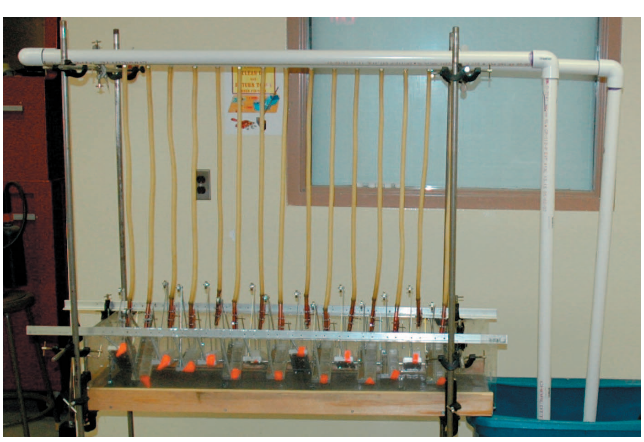

FIG. 2. (Color) A water circulation system, including blue reservoir, white pipes, and yellow hoses, powers the one-way coupled array of bistable oscillators. Video of the apparatus in operation is available online [14] and shows the solitons propagating right to left.

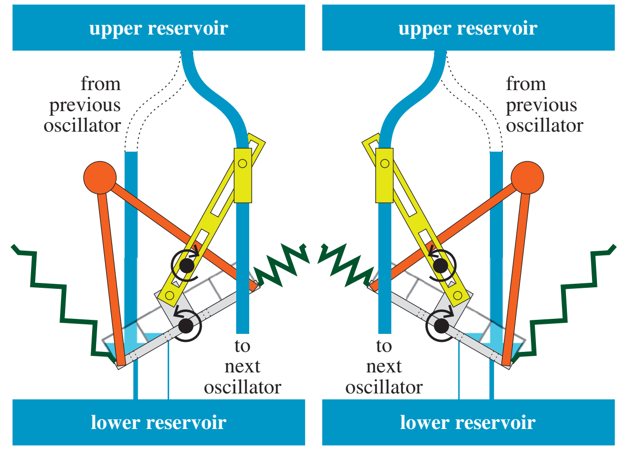


FIG. 3. (Color) A single oscillator, shown at two extremes of its motion, consists of an inverted pendulum (red) and two restoring springs (green), which together create a bistable element. A slotted strip (yellow) directs falling water (blue) into the next oscillator (not shown). Because of the fixed parallel axles (black), when the water rotates the pendulum one way, the slotted strip rotates the opposite way.

tended through the base allow water drainage. (The pressure of the water alone is sufficient in most cases to rotate the oscillators; the compartments provide an extra margin of torque against friction but do not noticeably alter the linearity of the coupling.)

Opposing 5.0 N/m springs attach to each oscillator, perpendicular to the axis of rotation. Metal washer counterweights at the top of the pendulum ensure bistability. When the pendulum tilts to one side, the gravitational torque on the pendulum top balances the combined spring torques, resulting in two stable equilibria on either side of the vertical.

Coupling is accomplished by directing a vertical water jet onto the next oscillator. Each oscillator rotates about a fixed central axle and is connected to a slotted strip that rotates (and slides) about a parallel fixed axle. The slotted strip controls the horizontal movement of the water jet. When an oscillator rotates one way, the slotted strip rotates (and slides) the other way, moving the jet nozzle to the opposite side. The falling water thus causes the next oscillator to rotate in the reverse direction.

To implement periodic boundary conditions, we split one oscillator into two parts, one with the pendulum and the other with the slotted strip, and install them at opposite ends of the array. The parts are connected by meshing gears and an auxiliary parallel axle (not shown in Fig. 3). This facilitates changing the length of the array to investigate the dramatic effects of parity on the dynamics.

A water circulation system powers the one-way coupled array. A 140 L reservoir stores water, which is pumped into both ends of a pressurized 3.2 cm polyvinyl chloride (PVC) pipe suspended about 1 m above the array parallel to its axles. A 1.25 L/s pump at one end of the pipe and a 2.50 L/s pump at the other provide sufficient pressure to evenly distribute water to latex tubes attached to the pipe via evenly spaced hose nipples. The water flows through the tubes, out of the nozzles, and onto the oscillators. As it drains

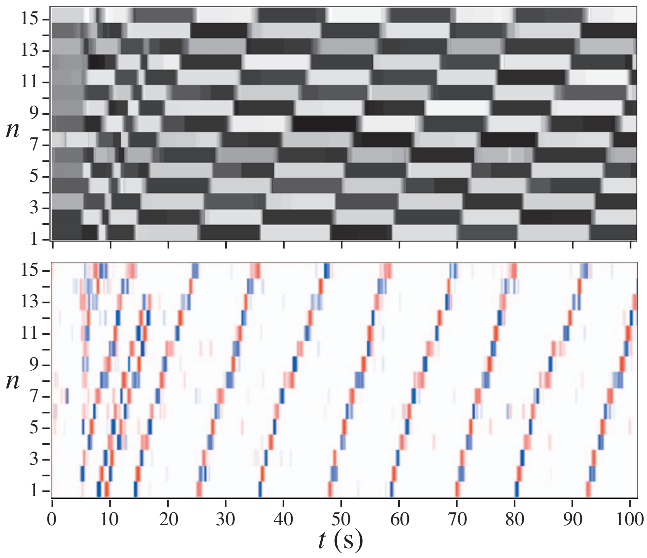


FIG. 4. (Color) Experimental spacetime evolution of a cyclic array of $N=15$ one-way coupled oscillators. Grays code angles $\theta_n[t]$ (top) and red-white-blue hues code negative-zero-positive angular velocities $\dot{\theta}_n[t]$ (bottom). From random initial conditions, a soliton pair annihilates at about $t=16$ s, leaving a single soliton to propagate indefinitely.

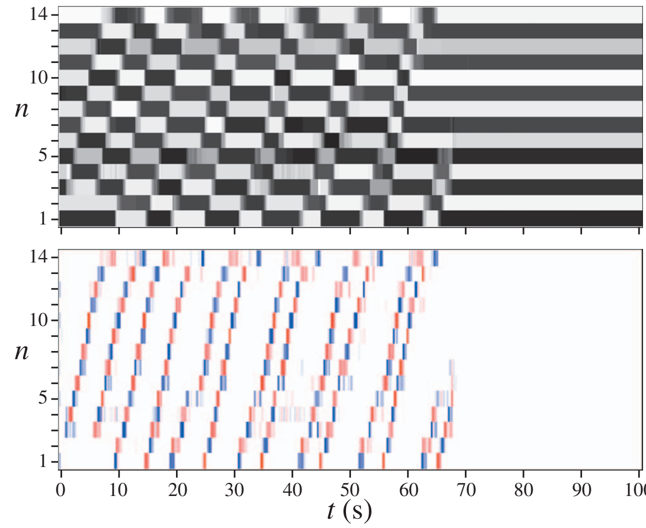


FIG. 5. (Color) Experimental spacetime evolution of a cyclic array of $N=14$ one-way coupled oscillators. Grays code angles $\theta_n[t]$ (top) and red-white-blue hues code negative-zero-positive angular velocities $\dot{\theta}_n[t]$ (bottom). From random initial conditions, a soliton pair annihilates at about $t=70$ s, leaving a quiescent array. (We stopped recording shortly thereafter and extrapolated the static spacetimes to emphasize the final static state and to better compare with Fig. 4.)

from the compartmented boxes, it falls onto a shelf that directs it back to the reservoir. A system of lab rods, stands, and clamps supports the entire apparatus, including the shelf and pressurized pipe. A detailed computer simulation of the exact apparatus vets the design before construction.

## IV. EXPERIMENT

We acquire data by videotaping the evolution of the array. We import the video to an iMac computer and analyze it using VIDEOPOINT [13] software to track the motion of the oscillators. Using elementary trigonometry, we infer the angle of each oscillator from its vertical displacement in the video.

We first configure the apparatus as a 15-element array. We initially set each oscillator in one stable equilibrium or the other. Flowing water couples the elements, resulting in multiple solitons that annihilate in pairs, leaving a single soliton to propagate indefinitely [14].

The top panel of Fig. 4 depicts a typical example of the spacetime evolution of the array, time rightward, oscillator number upward. Grays code angles $\theta_n[t]$. Most of the time, individual oscillators are at rest, rotated positively (white) or negatively (black). Movement occurs when a soliton passes through, reversing an oscillator's rotation. The bottom panel of Fig. 4 depicts the spacetime evolution of the angular velocities $\dot{\theta}_n[t]$, approximated by taking finite differences of the angles. Blues correspond to positive angular velocities, reds to negative. Color saturation is proportional to speed, so that white represents zero. In the top panel, solitons appear as diagonal discontinuities. In the bottom panel, solitons appear as alternating red and blue diagonal lines, marking successive reversals of oscillator rotation.

Quiescence is possible in even-element arrays, making annihilations more dramatic. Consequently, we next configure the apparatus as a 14-element array. After the array reaches its static, spatially alternating attractor, we introduce a soliton by manually reversing and holding a single oscillator. When the soliton travels the desired separation distance, we release the oscillator, inducing a second soliton. The two solitons travel through the array until they collide and annihilate.

Figure 5 illustrates a typical example. As before, the top panel depicts angles and the bottom panel depicts angular velocities. During this run, the gears implementing the periodic boundary conditions introduce sufficient friction to cause the last oscillator to reverse somewhat sluggishly.

In computer simulations of these systems, we find very long transients and annihilation times, which we want to investigate with this experimental system. Consequently, we systematically record the times to annihilate $t$, as a function of initial soliton separation $\Delta n$ in our 14-element array. We always measure the separation in the direction of propagation from the leading edge of the fiducial soliton (at one end of the array) to the leading edge of the other soliton. We call the smaller of $\Delta n$ and $N-\Delta n$ the *minimum* separation.

Figure 6 reports our results. On average, since annihilation depends on close approach, the larger the initial minimum separation of the solitons, the longer they take to annihilate. The larger minimum separations are also associated with bigger spreads in annihilation times. For an array of identical elements, the plot would be symmetric, because separations of $\Delta n$ and $N-\Delta n$ are physically equivalent. However, in our experimental apparatus, there are slight positive-negative asymmetries in the ease with which the oscillators reverse. As we show in Sec. V, this can lead to the observed asymmetry in Fig. 6.

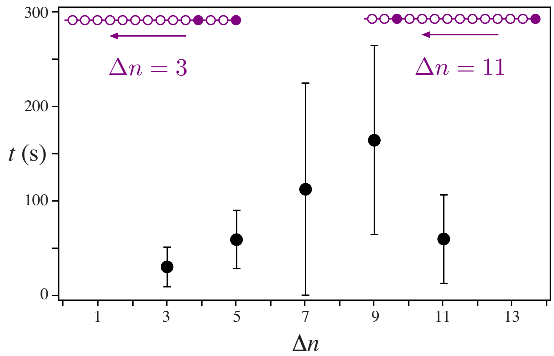


FIG. 6. (Color online) Experimental means and standard deviations of 100 soliton annihilation times $t$ as a function of initial separation $\Delta n$, for an array of $N=14$ oscillators, with 20 trials for each of 5 $\Delta n$. Pairs of solitons with different separations $\Delta n=3$ and 11 (insets), measured from the fiducial soliton (at one end of the array) in the direction of propagation (arrow), share the same *minimum* separation of 3.

## V. ANALYSIS

As the solitons propagate, they vibrate the array. This is a source of noise not included in Eq. (2). Consequently, to better understand our experiments, we perform computer simulations of one-way coupling in the presence of both disorder *and* noise. We generalize Eq. (2) to

$$I\ddot{\theta}_n = -\gamma_n[\dot{\theta}_n]\dot{\theta}_n + A\theta_n - b\theta_n^3 - \kappa\theta_{n-1} + \xi_n[t]. \tag{3}$$

We model the slight rotational asymmetry of the oscillators with asymmetric (or direction-dependent) viscosities

$$\gamma_n[\dot{\theta}_n] = \begin{cases} \gamma_{n+}, & \dot{\theta}_n > 0, \\ \gamma_{n-}, & \dot{\theta}_n \leq 0, \end{cases} \tag{4}$$

where $\gamma_{n\pm}$ are static, uniformly distributed random variables. We model the noise by taking $\xi_n[t]$ to be independent, exponentially correlated, Gaussian random processes. We choose colored noise because viscosity damps higher frequencies. We generate the colored noise using the Fox *et al.* algorithm [15] and numerically integrate the stochastic equations of motion using the Euler-Maruyama algorithm [16].

In the simulation, rotational inertia $I=1.0$. Spring and pendulum constants $a=0.72$ and $b=0.5$ shape the bistable potential. The coupling $\kappa=0.5$ and the mean viscosity $\gamma=1.0$. Three interrelated parameters characterize the colored noise: the noise intensity $D=0.01$, the correlation time $\tau=0.1$, and the variance $\sigma^2=D/\tau$. Our integration time step $dt=0.005$. Units of time are arbitrary. We distribute the calculations over the University of Portland's 32-processor Mac cluster.

We simulate the annihilation of pairs of solitons in an array of $N=14$ elements. Fixing the initial position of one soliton, we vary the initial position of the second soliton. For each separation $\Delta n$, we record the time to annihilation for 32 000 different realizations of the noise but the same realization of the disorder. Consistent with the experimental data of Fig. 6, the annihilation times have very broad distributions.

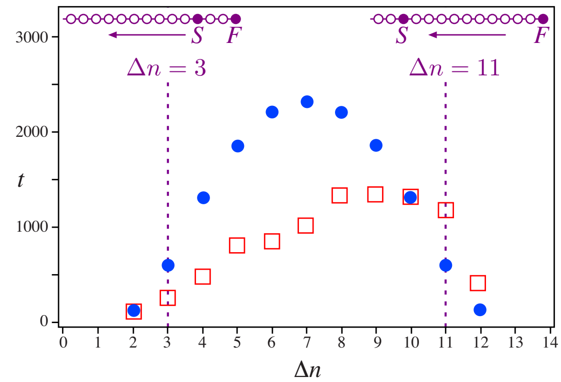


FIG. 7. (Color online) Simulated mean soliton annihilation times $t$ as a function of initial separation $\Delta n$, for an array of $N=14$ oscillators without (filled blue disks) and with (framed red boxes) 20% asymmetry in the oscillator viscosities. These asymmetries make soliton $F$ (inset) faster and soliton $S$ slower than otherwise. This speed differential advances the typical noise-induced proximity annihilation of $F$ by $S$ for $\Delta n=3$ but retards the typical noise-induced proximity annihilation of $S$ by $F$ for $\Delta n=11$.

We simulate many realizations of disorder and plot the mean annihilation time $t$ as a function of initial separation $\Delta n$. Generically, we find that asymmetric disorder skews the mean annihilation time plots. In Fig. 7, we present a particular realization (framed boxes) of 20% asymmetric disorder that generates a skewed annihilation time plot similar to the experimental data of Fig. 6. For comparison, we also present the symmetric case (filled circles). Table I reports the specific viscosities.

As two solitons chase each other through the array, they rotate each oscillator in opposite ways. The asymmetric, disordered viscosity can thereby make it harder or easier for a soliton to propagate. For example, if the viscosity makes it easier, on average, for the fiducial soliton $F$ to rotate the oscillators, it will tend to propagate faster than the other, slower soliton $S$, as indicated in the insets of Fig. 7. How-

TABLE I. Specific realization of 20% disorder used in Fig. 7.

| $n$ | $\gamma_{n-}$ | $\gamma_{n+}$ |
|---|---|---|
| 1 | 1.024552 | 0.889993 |
| 2 | 0.957237 | 0.977575 |
| 3 | 0.914017 | 0.857912 |
| 4 | 1.025422 | 1.145871 |
| 5 | 1.158161 | 0.892322 |
| 6 | 0.801605 | 0.984232 |
| 7 | 0.870529 | 1.050466 |
| 8 | 1.177949 | 1.138434 |
| 9 | 0.803999 | 0.902832 |
| 10 | 0.816301 | 0.977579 |
| 11 | 0.869777 | 0.946608 |
| 12 | 0.823528 | 1.024973 |
| 13 | 0.853576 | 1.151475 |
| 14 | 1.027378 | 1.008514 |

ever, annihilations still typically occur by noise-induced fluctuations, which are effective when the solitons are in close proximity. Thus, the speed differential will tend to advance the noise-induced proximity annihilations for small $\Delta n$ (left inset, Fig. 7, where $F$ will catch $S$ but more quickly than without the differential). Similarly, the speed differential will tend to retard the noise-induced proximity annihilations for large $\Delta n$ (right inset, Fig. 7, where $S$ will catch $F$ but not as quickly as without the differential).

Therefore, if the fiducial soliton experiences less viscosity than the other soliton, the annihilation time plot skews by decreasing times for smaller initial separations and increasing times for larger initial separations. Conversely, the annihilation time plot skews in the other direction if the fiducial soliton experiences more viscosity than the other soliton.

## VI. CONCLUSIONS

Formally, one-way coupling involves deleting one of two coupling terms from Eq. (1), which describes a well-studied array of bistable oscillators. This breaks the symmetry of the array and generates solitons that propagate one way in a damped medium that would otherwise suppress any excitation [4,8]. In practice, we exploit the fact that the downward force of a water jet is independent of the transverse force that directs it to realize the dynamics of Eq. (2) in a simple and inexpensive mechanical array.

As we have shown here, one-way coupling is not simply a mathematical exercise. Our experiments highlight the importance of both noise and disorder in understanding the dynamics of one-way coupled arrays. In particular, as elucidated by our simulations, the inevitable slight asymmetries in oscillator rotation lead to asymmetries in soliton speeds that skew annihilation time distributions. Such work illustrates the synergy that theory, simulation, and experiment can bring to understanding a different system. Early theory and simulations inspired our experiment, which in turn has inspired further simulations and theory.

One-way coupling is a special, extreme case of wave propagation in anisotropic media. Future work includes the study of transitional behavior as the coupling ranges from fully isotropic to fully anisotropic, a complete understanding of the role of noise (especially for very large $N$ arrays), and generalizations to higher dimensions where different phenomena are currently under investigation.

## ACKNOWLEDGMENTS

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[1] T. Dauxois and M. Payrard, *Physics of Solitons* (Cambridge University Press, Cambridge, UK, 2006).
[2] J. M. Carcione, *Wave Fields in Real Media: Wave Propagation in Anisotropic, Inelastic, Porous and Electromagnetic Media*, 2nd ed. (Elsevier Science, Amsterdam, 2007).
[3] A. A. Fuki, Y. A. Kravtsov, and O. N. Naida, *Geometrical Optics of Weakly Anisotropic Media* (CRC, Boca Raton, FL, 1998).
[4] V. In, A. R. Bulsara, A. Palacios, P. Longhini, A. Kho, and J. D. Neff, Phys. Rev. E **68**, 045102(R) (2003).
[5] A. R. Bulsara, V. In, A. Kho, P. Longhini, A. Palacios, W. J. Rappel, J. Acebron, S. Baglio, and B. Ando, Phys. Rev. E **70**, 036103 (2004).
[6] J. A. Rogge and D. Aeyels, J. Phys. A **37**, 11135 (2004).
[7] V. In, A. R. Bulsara, A. Palacios, P. Longhini, and A. Kho, Phys. Rev. E **72**, 045104(R) (2005).
[8] J. F. Lindner and A. R. Bulsara, Phys. Rev. E **74**, 020105(R) (2006).
[9] V. In, A. Palacios, A. R. Bulsara, P. Longhini, A. Kho, J. D. Neff, S. Baglio, and B. Ando, Phys. Rev. E **73**, 066121 (2006).
[10] A. R. Bulsara, J. F. Lindner, V. In, A. Kho, S. Baglio, V. Sacco, B. Ando, P. Longhini, A. Palacios, and W. J. Rappel, Phys. Lett. A **353**, 4 (2006).
[11] B. Razavi, *Fundamentals of Microelectronics* (Wiley, New York 2008).
[12] We use the term "soliton" in the general sense of a wavelike disturbance that propagates for long times while maintaining its shape.
[13] Lenox Softworks, Inc. VIDEOPOINT www.vpfundamentals.com.
[14] See EPAPS Document No. E-PLEEE8-78-041812 for a QuickTime movie of the 15-element array in operation. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.
[15] R. F. Fox, I. R. Gatland, R. Roy, and G. Vemuri, Phys. Rev. A **38**, 5938 (1988).
[16] P. E. Kloeden and E. Platen, *Numerical Solutions of Stochastic Differential Equations* (Springer, Berlin, 1992).